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The integrals in Gradshteyn and Ryzhik. Part 17: The Riemann zeta function

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Abstract. The table of Gradshteyn and Ryzhik contains some integrals that can be expressed in terms of the Riemann zeta $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$. In this note we present some of these evaluations.

1. Introduction

The table of integrals [**3**] contains a large variety of definite integrals that involve the *Riemann zeta* function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.1}$$

The series converges for $\operatorname{Re} s > 1$.

This is a classical function that plays an important role in the distribution of prime numbers. The reader will find in [**2**] a historical description of the fundamental properties of $\zeta(s)$. The textbook [**4**] presents interesting information about the major open question related to $\zeta(s)$: all its non-trivial zeros are on the vertical line $\operatorname{Re} s = \frac{1}{2}$. This is the famous *Riemann hypothesis*.

In this section we summarize elementary properties of ζ that will be employed in the evaluation of definite integrals.

The zeta function at the even integers. The values of $\zeta(s)$ at the *even* integers are given in terms of the *Bernoulli numbers* defined by the generating function

$$\frac{u}{e^u - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} u^k. \tag{1.2}$$

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It turns out that $B_{2n+1} = 0$ for $n > 1$. The relation

$$\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \tag{1.3}$$

can be found in [**1**]. The sign of B_{2n} is $(-1)^{n-1}$, so we can write (1.3) as

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} |B_{2n}|, \tag{1.4}$$

that looks more compact. The case of $\zeta(2n+1)$ is more compicated. No simple expression, such as (1.4), is known.

There are other series that can be expressed in terms of $\zeta(s)$. We present here the case of the alternating zeta series.

Proposition 1.1. Assume $s > 1$. Then

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = (2^{1-s} - 1)\zeta(s). \tag{1.5}$$

Proof. Split the sum (1.1) according to the parity of n. Then

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} &= \sum_{k=1}^{\infty} \frac{1}{(2k)^s} - \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} \\ &= 2^{-s} \sum_{k=1}^{\infty} \frac{1}{k^s} - \left(\sum_{k=1}^{\infty} \frac{1}{k^s} - \sum_{k=1}^{\infty} \frac{1}{(2k)^s} \right). \end{aligned}$$

The identity (1.5) has been established. □

Note 1.2. The expression (1.5), written as

$$\zeta(s) = \frac{1}{2^{1-s} - 1} \sum_{n=1}^{\infty} \frac{(-1)^k}{k^s} \tag{1.6}$$

provides a *continuation* of $\zeta(s)$ to $0 < \mathrm{Re}\, s$, with the natural exception at $s = 1$.

Proposition 1.3. Let $a > 1$. Then

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^a} = \frac{2^a - 1}{2^a} \zeta(a). \tag{1.7}$$

Proof. This simply comes from

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^a} = \sum_{k=1}^{\infty} \frac{1}{k^a} - \sum_{k=1}^{\infty} \frac{1}{(2k)^a}.$$

□

2. A first integral representation

The first integral in [**3**] that is evaluated in terms of the Riemann zeta function is **3.411.1**:

$$\int_0^\infty \frac{x^{s-1}\,dx}{e^{px}-1} = \frac{\Gamma(s)\zeta(s)}{p^s}. \tag{2.1}$$

Here Γ is the *gamma function* defined by

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt. \tag{2.2}$$

To verify (2.1) observe that the parameter p can be scaled out of the integral. Indeed, the change of variables $t = px$ shows that (2.1) is equivalent to the case $p = 1$:

$$\int_0^\infty \frac{t^{s-1}\,dt}{e^t-1} = \Gamma(s)\zeta(s). \tag{2.3}$$

To prove this, expand the integrand as

$$\frac{1}{e^t-1} = \frac{e^{-t}}{1-e^{-t}} = \sum_{k=0}^\infty e^{-(k+1)t}. \tag{2.4}$$

Therefore,

$$\int_0^\infty \frac{x^{s-1}\,dx}{e^x-1} = \sum_{k=0}^\infty \int_0^\infty t^{s-1}e^{-(k+1)t}\,dt. \tag{2.5}$$

The change of variables $v = (1+k)t$ yields the result.

Example 2.1. The evaluation of **3.411.2**:

$$\int_0^\infty \frac{x^{2n-1}\,dx}{e^{px}-1} = (-1)^{n-1}\left(\frac{2\pi}{p}\right)^{2n}\frac{B_{2n}}{4n} \tag{2.6}$$

can be reduced to the case $p = 1$ by the scaling $t = px$ and it follows from (1.3). Using (1.4) we write it as

$$\int_0^\infty \frac{x^{2n-1}\,dx}{e^x-1} = \frac{(2\pi)^{2n}}{4n}|B_{2n}|. \tag{2.7}$$

Example 2.2. The evaluation of **3.411.3**:

$$\int_0^\infty \frac{x^{s-1}\,dx}{e^{px}+1} = \frac{(1-2^{1-s})\Gamma(s)}{p^s}\zeta(s), \tag{2.8}$$

is first reduced, via $t = px$, to the case $p = 1$:

$$\int_0^\infty \frac{t^{s-1}\,dx}{e^t+1} = (1-2^{1-s})\Gamma(s)\zeta(s), \tag{2.9}$$

and this is evaluated expanding the integrand and integrating term by term to obtain

$$\int_0^\infty \frac{t^{s-1}}{e^t+1}\,dt = \frac{1}{\Gamma(s)}\sum_{k=0}^\infty \frac{(-1)^k}{(k+1)^s}. \tag{2.10}$$

The result now follows from (1.5).

Example 2.3. The special case $s = 2n$ in (2.8) yields

$$\int_0^\infty \frac{t^{2n-1}\,dt}{e^t+1} = (1-2^{1-2n})\frac{(2\pi)^{2n}}{4n}|B_{2n}|. \tag{2.11}$$

The integral **3.411.4**:

$$\int_0^\infty \frac{x^{2n-1}\,dx}{e^{px}+1} = (1-2^{1-2n})\left(\frac{2\pi}{p}\right)^{2n}\frac{|B_{2n}|}{4n}, \tag{2.12}$$

is reduced to (2.11) by the usual scaling.

3. Integrals involving partial sums of $\zeta(s)$

In this section we consider in a unified form a series of definite integrals in [**3**] whose values involve partial sums of the Riemann zeta function. We begin with the evaluation of **3.411.6**: expanding the integrand we obtain

$$\begin{aligned} \int_0^\infty \frac{x^{a-1}e^{-\beta x}}{1-\delta e^{-\gamma x}}\,dx &= \sum_{k=0}^\infty \delta^k \int_0^\infty x^{a-1}e^{-x(\beta+\gamma k)}\,dx \\ &= \frac{\Gamma(a)}{\gamma^a}\sum_{k=0}^\infty \delta^k\left(k+\frac{\beta}{\gamma}\right)^{-a}. \end{aligned} \tag{3.1}$$

The sum is identified as the *Lerch function* defined by

$$\Phi(z,s,v) = \sum_{n=0}^\infty (v+n)^{-s}z^n. \tag{3.2}$$

Therefore

$$\int_0^\infty \frac{x^{a-1}e^{-\beta x}\,dx}{1-\delta e^{-\gamma x}} = \frac{\Gamma(a)}{\gamma^a}\Phi\left(\delta,a,\beta/\gamma\right). \tag{3.3}$$

Integrals involving the Lerch Φ-function will be discussed in a future publication. Here we simply observe that **3.411.22**:

$$\int_0^\infty \frac{x^{p-1}\,dx}{e^{rx}-q} = \frac{\Gamma(p)}{r^p}\Phi(q,p,1) \tag{3.4}$$

follows directly from (3.1) after writing

$$\int_0^\infty \frac{x^{p-1}\,dx}{e^{rx}-q} = \int_0^\infty \frac{x^{p-1}e^{-rx}\,dx}{1-qe^{-rx}}. \tag{3.5}$$

We now discuss several special cases of (3.1).

Example 3.1. The case $\delta = 1$ in (3.1) is related to the *Hurwitz zeta function* defined by

$$\zeta(z,q) = \sum_{n=0}^\infty \frac{1}{(n+q)^z}. \tag{3.6}$$

Replacing $\delta = 1$ in (3.1) gives

$$\int_0^\infty \frac{x^{a-1}e^{-\beta x}}{1-e^{-\gamma x}}\,dx = \frac{\Gamma(a)}{\gamma^a}\zeta(a,\beta/\gamma). \tag{3.7}$$

This appears as **3.411.7**.

Example 3.2. We now consider the special case of (3.7) in which β/γ is a positive integer, say, $\beta = m\gamma$. Then we obtain

$$\int_0^\infty \frac{x^{a-1}e^{-m\gamma x}\,dx}{1-e^{-\gamma x}} = \frac{\Gamma(a)}{\gamma^a}\sum_{k=0}^\infty \frac{1}{(m+k)^a}. \tag{3.8}$$

Now observe that

$$\sum_{k=0}^\infty \frac{1}{(m+k)^a} = \sum_{k=1}^\infty \frac{1}{k^a} - \sum_{k=1}^{m-1}\frac{1}{k^a}, \tag{3.9}$$

so that

$$\int_0^\infty \frac{x^{a-1}e^{-m\gamma x}\,dx}{1-e^{-\gamma x}} = \frac{\Gamma(a)}{\gamma^a}\left(\zeta(a) - \sum_{k=1}^{m-1}\frac{1}{k^a}\right). \tag{3.10}$$

We restate the previous result.

Proposition 3.3. Let $a,\,\gamma \in \mathbb{R}^+$ and $m \in \mathbb{N}$. Then

$$\int_0^\infty \frac{x^{a-1}e^{-m\gamma x}\,dx}{1-e^{-\gamma x}} = \frac{\Gamma(a)}{\gamma^a}\left(\zeta(a) - \sum_{k=1}^{m-1}\frac{1}{k^a}\right). \tag{3.11}$$

Example 3.4. The value $a = 2$, $\gamma = 1$ and $m = 1$ in (3.10) give

$$\int_0^\infty \frac{xe^{-x}\,dx}{e^x-1} = \frac{\pi^2}{6} - 1, \tag{3.12}$$

suing $\Gamma(2) = 1$ and $\zeta(2) = \pi^2/6$. This appears as **3.411.9** in [**3**].

Example 3.5. The case $a = 3$, $\gamma = 1$ and $m \in \mathbb{N}$ gives **3.411.14**:

$$\int_0^\infty \frac{x^2e^{-mx}}{1-e^{-x}}\,dx = 2\left(\zeta(3) - \sum_{k=1}^{m-1}\frac{1}{k^3}\right). \tag{3.13}$$

Example 3.6. The case $a = 4$, $\gamma = 1$ and $m \in \mathbf{N}$ give **3.411.17**:

$$\int_0^\infty \frac{x^3e^{-mx}}{1-e^{-x}}\,dx = \frac{\pi^4}{15} - 6\sum_{k=1}^{m-1}\frac{1}{k^4}. \tag{3.14}$$

Here we have used $\Gamma(4) = 6$ and $\zeta(4) = \pi^4/90$.

Example 3.7. Formula **3.411.25** is:

$$\int_0^\infty x\,\frac{1+e^{-x}}{e^x-1}\,dx = \int_0^\infty \frac{xe^{-x}\,dx}{1-e^{-x}} + \int_0^\infty \frac{xe^{-2x}\,dx}{1-e^{-x}}. \tag{3.15}$$

The first integral corresponds to $a = 2$, $\gamma = 1$, $m = 1$ and the second one to $a = 2$, $\gamma = 1$, $m = 2$. Therefore

$$\int_0^\infty x\,\frac{1+e^{-x}}{e^x-1}\,dx = \Gamma(2)\left(\zeta(2)+\zeta(2)-1\right) = \frac{\pi^2}{3}-1. \tag{3.16}$$

Example 3.8. The final example in this section is **3.411.21**:

$$\int_0^\infty x^{n-1}\frac{1-e^{-mx}}{1-e^x}\,dx = (n-1)!\sum_{k=1}^m \frac{1}{k^n}. \tag{3.17}$$

We now show that the *correct formula* is

$$\int_0^\infty x^{n-1}\frac{1-e^{-mx}}{1-e^x}\,dx = -(n-1)!\sum_{k=1}^m \frac{1}{k^n}. \tag{3.18}$$

To establish this, we write

$$\int_0^\infty x^{n-1}\frac{1-e^{-mx}}{1-e^x}\,dx = \int_0^\infty \frac{x^{n-1}e^{-(m+1)x}}{1-e^{-x}}\,dx - \int_0^\infty \frac{x^{n-1}e^{-x}}{1-e^{-x}}\,dx. \tag{3.19}$$

The first integral corresponds to $a = n$, $\gamma = 1$ and $m+1$ instead of m, so that

$$\int_0^\infty \frac{x^{n-1}e^{-(m+1)x}}{1-e^{-x}}\,dx = \Gamma(n)\left(\zeta(n)-\sum_{k=1}^m \frac{1}{k^n}\right). \tag{3.20}$$

The second integral corresponds to $a = n$, $\gamma = 1$ and $m = 1$. Therefore

$$\int_0^\infty \frac{x^{n-1}e^{-x}}{1-e^{-x}}\,dx = \Gamma(n)\zeta(n). \tag{3.21}$$

Formula (3.18) has been established.

4. The alternating version

The alternating version of (3.1) gives

$$\int_0^\infty \frac{x^{a-1}e^{-\beta x}}{1+\delta e^{-\gamma x}}\,dx = \frac{\Gamma(a)}{\gamma^a}\sum_{k=0}^\infty (-1)^k\delta^k\left(k+\frac{\beta}{\gamma}\right)^{-a}, \tag{4.1}$$

that in the case $\delta = 1$ provides

$$\int_0^\infty \frac{x^{a-1}\,e^{-\beta x}\,dx}{1+e^{-\gamma x}} = \frac{\Gamma(a)}{\gamma^a}\sum_{k=0}^\infty (-1)^k(k+\beta/\gamma)^{-a}. \tag{4.2}$$

In particular, if $\beta = m\gamma$, with $m \in \mathbb{N}$, we have

$$\int_0^\infty \frac{x^{a-1}e^{-m\gamma x}\,dx}{1+e^{-\gamma x}} = \frac{\Gamma(a)}{\gamma^a}\sum_{k=0}^\infty \frac{(-1)^k}{(k+m)^a}. \tag{4.3}$$

Using (1.5) we obtain the next proposition:

Proposition 4.1. Let $a, \gamma \in \mathbb{R}^+$ and $m \in \mathbb{N}$. Then

$$\int_0^\infty \frac{x^{a-1}e^{-m\gamma x}\,dx}{1+e^{-\gamma x}} = \frac{(-1)^m\Gamma(a)}{\gamma^a}\left((2^{1-a}-1)\zeta(a) - \sum_{k=1}^{m-1}\frac{(-1)^k}{k^a}\right). \tag{4.4}$$

The next examples come from (4.3).

Example 4.2. The case $a = n$, $\gamma = 1$ and $m = p+1$ give **3.411.8**:

$$\int_0^\infty \frac{x^{n-1}e^{-px}\,dx}{1+e^{x}} = (-1)^p\Gamma(n)\left[(1-2^{1-n})\zeta(n) + \sum_{k=1}^{p}\frac{(-1)^k}{k^n}\right]. \tag{4.5}$$

The reader will check that the answer can be written as

$$\int_0^\infty \frac{x^{n-1}e^{-px}\,dx}{1+e^{-x}} = (n-1)!\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(p+k)^n}. \tag{4.6}$$

Example 4.3. The case $a = 2$, $c = 1$ and $m = 2$ gives **3.411.10**:

$$\int_0^\infty \frac{xe^{-2x}}{1+e^{-x}}\,dx = 1 - \frac{\pi^2}{12}. \tag{4.7}$$

Example 4.4. The case $a = 2$, $c = 1$ and $m = 3$ gives **3.411.11**:

$$\int_0^\infty \frac{xe^{-3x}}{1+e^{-x}}\,dx = \frac{\pi^2}{12} - \frac{3}{4}. \tag{4.8}$$

Example 4.5. The case $a = 2$, $c = 1$ and $m = 2n$ gives **3.411.12**:

$$\int_0^\infty \frac{xe^{-(2n-1)x}}{1+e^{-x}}\,dx = -\frac{\pi^2}{12} + \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k^2}. \tag{4.9}$$

Example 4.6. The case $a = 2$, $c = 1$ and $m = 2n+1$ gives **3.411.13**:

$$\int_0^\infty \frac{xe^{-2nx}}{1+e^{-x}}\,dx = \frac{\pi^2}{12} + \sum_{k=1}^{2n}\frac{(-1)^k}{k^2}. \tag{4.10}$$

Example 4.7. The case $a = 3$, $c = 1$ and $m \in \mathbb{N}$ gives **3.411.15**:

$$\int_0^\infty \frac{x^2e^{-nx}}{1+e^{-x}}\,dx = (-1)^{n+1}\left(\frac{3}{2}\zeta(3) + 2\sum_{k=1}^{n-1}\frac{(-1)^k}{k^3}\right). \tag{4.11}$$

Example 4.8. The case $a = 4$, $c = 1$ and $m \in \mathbb{N}$ gives **3.411.18**:

$$\int_0^\infty \frac{x^3e^{-nx}}{1+e^{-x}}\,dx = (-1)^{n+1}\left(\frac{7\pi^4}{120} + 6\sum_{k=1}^{n-1}\frac{(-1)^k}{k^4}\right). \tag{4.12}$$

Example 4.9. Similar manipulations produces **3.411.26**:

$$\int_0^\infty xe^{-x}\frac{1-e^{-x}}{1+e^{-3x}}\,dx = \frac{2\pi^2}{27}. \tag{4.13}$$

5. The logarithmic scale

The integrals described in Section 4 can be transformed into logarithmic integrals via the change of variables $t = e^{-cx}$. For example (3.1) becomes

$$\int_0^1 \frac{t^{\beta-1} \ln^{a-1} t \, dt}{1-\delta t} = (-1)^{a-1}\Gamma(a) \sum_{k=0}^{\infty} \frac{\delta^k}{(k+\beta)^a} \tag{5.1}$$

and the special case $\delta = 1$ replaces (3.7) with

$$\int_0^1 \frac{t^{\beta-1} \ln^{a-1} t \, dt}{1-t} = (-1)^{a-1}\Gamma(a) \sum_{k=0}^{\infty} \frac{1}{(k+\beta)^a}. \tag{5.2}$$

In the special case that $m \in \mathbb{N}$, the formula (3.11) becomes

$$\int_0^1 \frac{t^{m-1} \ln^{a-1} t \, dt}{1-t} = (-1)^{a-1}\Gamma(a) \left(\zeta(a) - \sum_{k=1}^{m-1} \frac{1}{k^a} \right), \tag{5.3}$$

in particular, for $m = 1$, we have

$$\int_0^1 \frac{\ln^{a-1} t \, dt}{1-t} = (-1)^{a-1}\Gamma(a)\zeta(a). \tag{5.4}$$

Finally, the change of variables $t = s^\gamma$ in (5.2) produces

$$\int_0^1 \frac{s^{\beta-1} \ln^{a-1} s \, dt}{1-s^\gamma} = (-1)^{a-1}\Gamma(\gamma) \sum_{k=0}^{\infty} \frac{1}{(\gamma k+\beta)^a}. \tag{5.5}$$

We now present examples of these formulas that appear in [**3**].

Example 5.1. Formula (5.4) appears in [**3**] only for a even. This is the case where the value of $\zeta(a)$ reduces via (1.3). We find **4.231.2** for $a = 2$:

$$\int_0^1 \frac{\ln x \, dx}{1-x} = -\frac{\pi^2}{6}, \tag{5.6}$$

and **4.262.2**:

$$\int_0^1 \frac{\ln^3 x \, dx}{1-x} = -\frac{\pi^4}{15}, \tag{5.7}$$

that uses $\Gamma(4) = 6$ and $\zeta(4) = \pi^4/90$. The next example is **4.264.2**:

$$\int_0^1 \frac{\ln^5 x \, dx}{1-x} = -\frac{8\pi^6}{63}, \tag{5.8}$$

that uses $\Gamma(6) = 120$ and $\zeta(6) = \pi^6/945$. The final example is **4.266.2**:

$$\int_0^1 \frac{\ln^7 x \, dx}{1-x} = -\frac{8\pi^8}{15}, \tag{5.9}$$

that uses $\Gamma(8) = 5040$ and $\zeta(8) = \pi^8/9450$.

Example 5.2. The choice $a = 4$ and $m = n + 1$ in (5.3) produces **4.262.5**:

$$\int_0^1 \frac{x^n \ln^3 x}{1-x}\, dx = -\frac{\pi^4}{15} + 6\sum_{k=1}^{n} \frac{1}{k^4}. \tag{5.10}$$

Example 5.3. The choice $a = 4$, $\beta = 2n + 1$, and $\gamma = 2$ in (5.5) gives **4.262.6**:

$$\int_0^1 \frac{x^{2n} \ln^3 x}{1-x^2}\, dx = -\frac{\pi^4}{16} + 6\sum_{k=1}^{n} \frac{1}{(2k+1)^4}. \tag{5.11}$$

In this calculation we have used (1.7) to produce the value

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}. \tag{5.12}$$

Example 5.4. The choice $a = 3$ and $m = n + 1$ in (5.3) gives **4.261.12**:

$$\int_0^1 \frac{x^n \ln^2 x}{1-x}\, dx = 2\left(\zeta(3) - \sum_{k=1}^{n} \frac{1}{k^3}\right). \tag{5.13}$$

Example 5.5. The choice $a = 3$, $\beta = 2n + 1$, and $\gamma = 2$ gives **4.261.13**:

$$\int_0^1 \frac{x^{2n} \ln^2 x}{1-x^2}\, dx = \frac{7\zeta(3)}{4} - 2\sum_{k=0}^{n-1} \frac{1}{(2k+1)^3}. \tag{5.14}$$

6. The alternating logarithmic scale

There is a corresponding list of formulas for logarithmic integrals that produce alternating series. For example (5.1) becomes

$$\int_0^1 \frac{t^{\beta-1} \ln^{a-1} t\, dt}{1+\delta t} = (-1)^{a-1}\Gamma(a) \sum_{k=0}^{\infty} \frac{(-1)^k \delta^k}{(k+\beta)^a} \tag{6.1}$$

and the case $\delta = 1$ gives

$$\int_0^1 \frac{t^{\beta-1} \ln^{a-1} t\, dt}{1+t} = (-1)^{a-1}\Gamma(a) \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+\beta)^a}. \tag{6.2}$$

In the special case that $m \in \mathbb{N}$, we have

$$\int_0^1 \frac{t^{m-1} \ln^{a-1} t\, dt}{1+t} = (-1)^{a+m}\Gamma(a) \left(\frac{2^{a-1}-1}{2^{a-1}}\zeta(a) + \sum_{k=1}^{m-1} \frac{(-1)^k}{k^a} \right), \tag{6.3}$$

in particular, for $m = 1$, we have

$$\int_0^1 \frac{\ln^{a-1} t\, dt}{1+t} = (-1)^{a+1} \frac{2^{a-1}-1}{2^{a-1}} \Gamma(a)\zeta(a). \tag{6.4}$$

Finally (5.5) produces

$$\int_0^1 \frac{s^{\beta-1} \ln^{a-1} s\, ds}{1+s^\gamma} = (-1)^{a-1}\Gamma(a) \sum_{k=0}^{\infty} \frac{(-1)^k}{(\gamma k+\beta)^a}. \tag{6.5}$$

We now present examples of these formulas that appear in [**3**].

Example 6.1. The choice $a = 2$ in (6.4) produces **4.231.1**:

$$\int_0^1 \frac{\ln x}{1+x}\,dx = -\frac{\pi^2}{12}. \tag{6.6}$$

The table contains formulas that use (6.4) only for a even, in that form, the integrals are expressible as powers of π. For example, **4.262.1**:

$$\int_0^1 \frac{\ln^3 x}{1+x}\,dx = -\frac{7\pi^4}{120}, \tag{6.7}$$

using $\Gamma(4) = 6$ and $\zeta(4) = \pi^4/90$. Similarly, **4.264.1**:

$$\int_0^1 \frac{\ln^5 x}{1+x}\,dx = -\frac{31\pi^6}{252} \tag{6.8}$$

uses $\Gamma(6) = 120$ and $\zeta(6) = \pi^6/945$. The final example of this form is **4.266.1**:

$$\int_0^1 \frac{\ln^7 x}{1+x}\,dx = -\frac{127\pi^8}{240}, \tag{6.9}$$

that employs $\Gamma(8) = 5040$ and $\zeta(8) = \pi^8/9450$. The next cases in this list would be

$$\int_0^1 \frac{\ln^9 x}{1+x}\,dx = -\frac{511\pi^{10}}{132}, \tag{6.10}$$

and

$$\int_0^1 \frac{\ln^{11} x}{1+x}\,dx = -\frac{1414477\pi^{12}}{32760}, \tag{6.11}$$

that do not appear in [**3**].

Example 6.2. The choice $a = 2n+1$ in (6.4) gives **4.271.1**:

$$\int_0^1 \frac{\ln^{2n} x}{1+x}\,dx = \frac{2^{2n}-1}{2^{2n}}(2n)!\,\zeta(2n+1). \tag{6.12}$$

Example 6.3. The choice $a = 2n$ in (6.4) gives **4.271.2**:

$$\int_0^1 \frac{\ln^{2n-1} x}{1+x}\,dx = -\frac{2^{2n-1}-1}{2^{2n-1}}(2n-1)!\,\zeta(2n), \tag{6.13}$$

and using (1.3) gives

$$\int_0^1 \frac{\ln^{2n-1} x}{1+x}\,dx = -\frac{2^{2n-1}-1}{2n}|B_{2n}|\pi^{2n}. \tag{6.14}$$

7. Integrals over the whole line

The change of variables $x = \frac{1}{p}e^{-t}$ in (2.1) gives entry **3.333.1**:

$$\int_{-\infty}^{\infty} \frac{e^{-sx}\,dx}{\exp(e^{-x})-1} = \Gamma(s)\zeta(s). \tag{7.1}$$

The same change of variable in (2.8) gives entry **3.333.2**:

$$\int_{-\infty}^{\infty} \frac{e^{-sx}\,dx}{\exp(e^{-x})+1} = (1-2^{1-s})\Gamma(s)\zeta(s). \tag{7.2}$$

The exceptional case

$$\int_{-\infty}^{\infty} \frac{e^{-x}\,dx}{\exp(e^{-x})+1} = \ln 2 \tag{7.3}$$

mentioned in entry **3.333.2**, is elementary.

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